Constructing zero divisors in the higher dimensional Cayley-Dickson algebras

Guillermo Moreno

Abstract: In this paper we give methods to construct zero divisors in the Cayley–Dickson algebras $\mathbb{A}_n = \mathbb{R}^{2^n}$ for n larger than 4. Also we relate the set of zero divisors with suitable Stiefel manifolds.

Introduction:

In this paper we return to the subject of 97' authors paper [M1] about the description of the zero divisors in the Cayley-Dickson algebras over the real numbers.

This paper must be seen as a sequel of the papers [M1] and [M2] and related to papers [M3] and [M4].

As we know, $\mathbb{A}_n = \mathbb{R}^{2^n}$ denotes the Cayley Dickson (C-D)-algebras over the real numbers. For $n = 0, 1, 2, 3$ are known as the clasical C-D algebra which correspond to $\mathbb{A}_0 = \mathbb{R}, \mathbb{A}_1 = \mathbb{C}, \mathbb{A}_2 = \mathbb{H}$ and $\mathbb{A}_3 = \emptyset$, Real, Complex, Quaternions and Octonions number and because they are normed and alternative algebras they lack of zero divisors.[Sch2] [K-Y].

For $n \geq 4, \mathbb{A}_n$ is no normed, non-alternative (but flexible) algebra and has zero divisors i.e.; there are nonzero elements x and y such that $xy = 0$.

By definition a zero divisor is a nonzero element a such that there exists nonzero b with $ab = 0$.

In [M1] the zero divisors in $\mathbb{A}_4$ and the zero divisors in $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$ with alternative coordinates are described.

For $n = 4$ the set of zero divisors in $\mathbb{A}_4$ of fixed norm can be identified with $V_{7,2}$ the real Stiefel Manifold of two frames in $\mathbb{R}^7$ and the singular set of (x, y) with $xy = 0$ and $||x|| = ||y|| = 1$ is homoemorphic to G_2 the exceptional simple Lie group of rank 2.See also [K-Y].

The description of the zero divisors in $\mathbb{A}_4$ is given by the known fibration

$$G_2 \xrightarrow{\pi} V_{7,2}$$

with fiber S^3 since all the nontrivial annihilators are 4 dimensional.

For $n \geq 5$ there is NO analogous description. We will show that the zero divisors are in $\mathbb{A}_{n+1}$ and $V_{2^n-1,2}$ are related, but they are not equal and the corresponding singular set has (unknown) complicated description. See [M3].

In §1 we recall the basic notation and theorems already proved in [M1], [M2] which are necesary for further results.

In §2 we study the basic facts of the linear operators left and right mutiplication by a fixed (pure) element, as in [M2], and then, we define a suitable $O(2)$-action on the double pure elements of $\mathbb{A}_n$.

In §3 we define the *Spectrum* of a non-zero double pure element looking at the structure of the linear operators defined by left and right multiplication by the element.

By definition, the Spectrum is a set of $(2^{n-2}-1)$ non negative real numbers attached to each nonzero double pure element in $\mathbb{A}_n$ for $n \geq 3$ and the presence of zero in the spectrum determines that the element is a zero divisor.

As consequence we see how big can be the annihilator of an element; this result complements very nicely with the recent results in [B-D-I].

Also we look at the $O(2)$-action on the doubly pure elements showing that: the elements in the same $O(2)$-orbit have the same spectrum and consequently the annihilators of the elements in the $O(2)$-orbit are all equal.

In §4 we study the zero divisors and we give new methods to construct many of them, noticing that for n larger than 4 the problem becomes extremely difficult.

Also we prove that:All non-zero pure element in $\mathbb{A}_n$ is a component of a zero divisor in $\mathbb{A}_{n+1}$

In §5 we study the relationship between the Stiefel manifold $V_{2^n-1,2}$ [J] and the set of zero divisors in $\mathbb{A}_{n+1}$ noticing that for n larger than 4 they are realted but are "very far" to be equal.

Some of the results of this paper were partially done in a personal manuscript by the author of 2001,(which had restricted circulation) and some also can be found in [Ch].

We emphasize the initial input and permanent support to study this subject given by Professor Fred Cohen with whom we are very grateful.

1. Basic definitions and lemmas.

Here $e_0 = (e_0, 0)$ is the algebra unit in $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1} = \mathbb{A}_n$.

Definition: $e_{2^{n-1}} = (0, e_0) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1} = \mathbb{A}_n$ is *the sympletic unit* and we denote it by $\widetilde{e}_0 = (0, e_0)$. For

$$a = (a_1, a_2) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1} = \mathbb{A}_n$$

$$\widetilde{a} := (-a_2, a_1) = (a_1, a_2)(0, e_0) = a\widetilde{e}_0.$$

is the complexification of a.

Because the right multiplication by $\widetilde{e}_0$.

$$R_{\widetilde{e}_0} : \mathbb{A}_n \longrightarrow \mathbb{A}_n$$

has the matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ in $M_{2^{n-1}}(\mathbb{R})$ in the canonical basis.

Notice that $\widetilde{\widetilde{a}} = -a$ for all $a \in \mathbb{A}_n$; *the trace* on $\mathbb{A}_n$ is the linear transformation

$$t_n : \mathbb{A}_n \longrightarrow \mathbb{A}_0 = \mathbb{R}$$

given by $t_n(a) = a + \overline{a} = 2$ (Real part of a). For $a = (a_1, a_2) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1} = \mathbb{A}_n$

$$t_n(a) = t_{n-1}(a_1).$$

Definition: An element a *is pure* in $\mathbb{A}_n$ if

$$t_n(a) = t_{n-1}(a_1) = 0$$

Notice that $a \in \mathbb{A}_n$ is pure if and only if $a \perp e_0$ and $a \perp b$ if and only if $ab = -ba$ for a and b pure elements in $\mathbb{A}_n$.

Definition: An element $a = (a_1, a_2) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1} = \mathbb{A}_n$ is *doubly pure* if it is pure i.e. $t_n(a) = 0$ and also $t_n(\widetilde{a}) = -t_{n-1}(a_2) = 0$.

Notice that a is doubly pure if and only if $a \in \{e_0, \widetilde{e}_0\}^{\perp}$.

Notation: ${}_0\mathbb{A}_n = \mathbb{R}^{2^n - 1} = \{e_0\}^{\perp}$ pure elements in $\mathbb{A}_n$. $\widetilde{\mathbb{A}}_n = \mathbb{R}^{2^n - 2} = \{e_0, \widetilde{e}_0\}^{\perp}$ doubly pure elements in $\mathbb{A}_n$.

Lemma 1.1. For a and b in $\widetilde{\mathbb{A}_n}$ we have that

1) $a\widetilde{e}_0 = \widetilde{a}$ and $\widetilde{e}_0 a = -\widetilde{a}$

2) $a\widetilde{a} = -||a||^2\widetilde{e}_0$ and $\widetilde{a}a = ||a||^2\widetilde{e}_0$ so $a \perp \widetilde{a}$

3) $\widetilde{a}b = -\widetilde{ab}$ with a pure element.

4) $a \perp b$ if and only if $\widetilde{a}b + \widetilde{b}a = 0$

5) $\widetilde{a} \perp b$ if and only if $ab = \widetilde{b}\widetilde{a}$

6) $\widetilde{a}b = a\widetilde{b}$ if and only if $a \perp b$ and $\widetilde{a} \perp b$.

Proof: Notice that a is pure if $\overline{a} = -a$ and if $a = (a_1, a_2)$ is doubly pure then $\overline{a}_1 = -a_1$ and $\overline{a}_2 = -a_2$.

1) $\widetilde{e}_0 a = (0, e_0)(a_1, a_2) = (-\overline{a}_2, \overline{a}_1) = (a_2, -a_1) = -(-a_2, a_1) = -\widetilde{a}$.

2) $a\widetilde{a} = (a_1, a_2)(-a_2, a_1) = (-a_1a_2 + a_1a_2, a_1^2 + a_2^2) = -||a||^2\widetilde{e}_0$

   $\widetilde{a}a = (-a_2, a_1)(a_1, a_2) = (-a_2a_1 + a_2a_1, -a_2^2 - a_1^2) = ||a||^2\widetilde{e}_0$.Now since $-2\langle\widetilde{a}, a\rangle = a\widetilde{a} + \widetilde{a}a = 0$ we have $a \perp \widetilde{a}$.

3) $\widetilde{a}b = (-a_2, a_1)(b_1, b_2) = (-a_2b_1 + b_2a_1, -b_2a_2 - a_1b_1)$. So $\widetilde{\widetilde{a}b} = (a_1b_1 + b_2a_2, b_2a_1 - a_2b_1) = (a_1, a_2)(b_1, b_2) = ab$ then $-\widetilde{a}b = \widetilde{\widetilde{\widetilde{a}b}} = \widetilde{ab}$.

   Notice that in this proof we only use that $\overline{a}_1 = -a_1$ i.e a is pure and b doubly pure.

4) $a \perp b \Leftrightarrow ab + ba = 0 \Leftrightarrow ab = -ba \Leftrightarrow \widetilde{ab} = -\widetilde{ba}$

   $\Leftrightarrow -\widetilde{a}b = \widetilde{b}a \Leftrightarrow \widetilde{a}b + \widetilde{b}a = 0$ by (3).

5) $\widetilde{a} \perp b \Leftrightarrow \widetilde{\widetilde{a}}b + \widetilde{b}\widetilde{a} = 0$ (by (4)) $\Leftrightarrow -ab + \widetilde{b}\widetilde{a} = 0$.

6) If $\widetilde{a} \perp b$ and $a \perp b$ then

$$\widetilde{a}b = -\widetilde{ab} = \widetilde{ba} = -\widetilde{b}a = a\widetilde{b}.$$

   Conversly, put $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ and define $c := (a_1b_1 + b_2a_2)$ and $d := (b_2a_1 - a_2b_1)$ in $\mathbb{A}_{n-1}$.

   Then $a\widetilde{b} = (a_1, a_2)(-b_2, b_1) = (-a_1b_2 + b_1a_2, b_1a_1 + a_2b_2)$ so $a\widetilde{b} = (-\overline{d}, \overline{c})$.

Now $ab = (a_1, a_2)(b_1, b_2) = (a_1b_1+b_2a_2, b_2a_1-a_2b_1) = (c, d)$ then $\widetilde{ab} = (-d, c)$ so $\widetilde{a}b = (d, -c)$. Suppose that $a\widetilde{b} = \widetilde{a}b$ so $\overline{c} = -c$ and $d = -\overline{d}$ then

$$\begin{aligned} t_n(ab) &= t_{n-1}(c) = c + \overline{c} = 0 \text{ and } a \perp b \\ t_n(\widetilde{a}b) &= t_{n-1}(d) = d + \overline{d} = 0 \text{ and } \widetilde{a} \perp b. \end{aligned}$$

Q.E.D.

Corollary 1.2 For each $a \neq 0$ in $\widetilde{\mathbb{A}}_n$. The fourth dimensional vector subspace generated by $\{e_0, \widetilde{a}, a, \widetilde{e}_0\}$ is a copy of $\mathbb{A}_2 = \mathbb{H}$ the Hamilton quaternions. We denote it by $\mathbb{H}_a$.

Proof: We suppose that $||a|| = 1$, otherwise we take $\frac{a}{||a||}$. Construct the following multiplication table

	e_0	$\widetilde{a}$	a	$\widetilde{e}_0$
e_0	e_0	$\widetilde{a}$	a	$\widetilde{e}_0$
$\widetilde{a}$	$\widetilde{a}$	$-e_0$	$+\widetilde{e}_0$	$-a$
a	a	$-\widetilde{e}_0$	$-e_0$	$\widetilde{a}$
$\widetilde{e}_0$	a	$\widetilde{e}_0$	$-\widetilde{a}$	$-e_0$

By lemma 1.1. $a\widetilde{e}_0 = \widetilde{a}$; $\widetilde{e}_0 a = -\widetilde{a}$; $\widetilde{a}\widetilde{e}_0 = \widetilde{\widetilde{a}} = -a$; $\widetilde{e}_0\widetilde{a} = -\widetilde{\widetilde{a}} = a$; $a\widetilde{a} = -\widetilde{e}_0$ and $\widetilde{a}a = \widetilde{e}_0$.

This multiplication table is the one of the Hamilton quaternions identifying $e_0 \leftrightarrow 1$, $\widetilde{a} \leftrightarrow \hat{\imath}$, $a \leftrightarrow \hat{\jmath}$ and $\widetilde{e}_0 \leftrightarrow \hat{k}$.

Q.E.D.

Artin theorem [Sch1] said that: "Any two elements in $\mathbb{A}_3 = \mathbb{O}$ generate an associative subalgebra".

Corollary 1.2 is an analogous to this for $\mathbb{A}_n$ with $n \geq 4$.

2. Left and Right multiplication.

For x and y in $\mathbb{A}_n$ $\langle x,y\rangle$ denotes the standard Euclidean inner product that in terms of the Cayley-Dickson multiplication is given by

$$t_n(x\overline{y}) = x\overline{y} + y\overline{x} = 2\langle x,y\rangle.$$

In [1] and [8] is proved that

$$\begin{aligned}\langle ax,b\rangle &= \langle x,\overline{a}b\rangle \qquad \text{and}\\ \langle a,xb\rangle &= \langle a\overline{b},x\rangle\end{aligned}$$

for all a,b and x in $\mathbb{A}_n$.

Therefore if L_a and $R_b : \mathbb{A}_n \to \mathbb{A}_n$ denotes the linear transformation left and right multiplication by a and b respectively then $L_a^T = L_{\overline{a}}$ and $R_b^T = R_{\overline{b}}$.

So for a and b pure elements $L_a^T = -L_a$ and $R_b^T = -R_b$ i.e. L_a and R_b are skew–symmetric linear transformations then $(-L_a^2)$ and $(-R_b^2)$ are symmetric definite nonegatives linear transformations.

Lemma 2.1 For a a doubly pure element in $\mathbb{A}_n$, $n \geq 3$

$$R_a R_{\widetilde{e_0}} + R_{\widetilde{e_0}} R_a = 0 \quad \text{and} \quad L_a L_{\widetilde{e_0}} + L_{\widetilde{e_0}} L_a = 0.$$

Proof: Using Lemma 1.1 (3).

$(R_a R_{\widetilde{e_0}} + R_{\widetilde{e_0}} R_a)(x) = (x\widetilde{e}_0)a + (xa)\widetilde{e}_0 = \widetilde{x}a + \widetilde{xa} = 0$ for $x \in {}_0\mathbb{A}_n$. If $x = re_0$ for $r \in \mathbb{R}$ $\widetilde{x}a + \widetilde{xa} = r\widetilde{e}_0 a + (\widetilde{re_0})a = r(-\widetilde{a} + \widetilde{a}) = 0$. Then $(R_a R_{\widetilde{e_0}} + R_{\widetilde{e_0}} R_a)(x) = 0$ for all $x \in \mathbb{A}_n$.

On the other hand we have that for $y = (y_1, y_2)$

$$\widetilde{e}_0 y = (0, e_0)(y_1, y_2) = (-\overline{y}_2, \overline{y}_1)$$

then

$$\begin{aligned}\widetilde{e}_0(ax) &= (0,e_0)(a_1x_1 - \overline{x}_2a_2, x_2a_1 + a_2\overline{x}_1)\\ &= (-\overline{(x_2a_1 + a_2\overline{x}_1)}, \overline{(a_1x_1 - \overline{x}_2a_2)})\\ &= (-(\overline{a}_1\overline{x}_2 + x_1\overline{a}_2), (\overline{x}_1\overline{a}_1 - \overline{a}_2x_2)) \quad \text{and because } a \in \widetilde{\mathbb{A}_n}\\ &= (a_1\overline{x}_2 + x_1a_2, -\overline{x}_1a_1 + a_2x_2) \quad \text{for } x = (x_1,x_2) \in \mathbb{A}_n\end{aligned}$$

and $a(\widetilde{e}_0 x) = (a_1, a_2)(-\overline{x}_2, \overline{x}_1) = (-a_1\overline{x}_2 - x_1 a_2, \overline{x}_1 a_1 - a_2\overline{x}_2)$ therefore $(L_{\widetilde{e}_0} L_a + L_a L_{\widetilde{e}_0})(x) = \widetilde{e}_0(ax) + a(\widetilde{e}_0 x) = 0$ for $x \in \mathbb{A}_n$

Q.E.D.

Theorem 2.2. For $a \in \mathbb{A}_n$ pure element $L_a^2 = R_a^2$.

Proof: (Case doubly pure). For $a \in \widetilde{\mathbb{A}}_n$, $\mathbb{A}_n = \mathbb{H}_a \oplus \mathbb{H}_a^{\perp}$ so if $x \in \mathbb{H}_a$ then (because $\mathbb{H}_a$ is associative)

$$a(ax) = a^2 x = -||a||^2 x = -x||a||^2 = xa^2 = (xa)a.$$

If $x \in \mathbb{H}^{\perp}$ then $a \perp x$ and by flexibility $a(ax) = -a(xa) = -(ax)a = (xa)a$ so $L_a^2 = R_a^2$.

(General case). Suppose that $a \in \widetilde{\mathbb{A}}_n$ and $r \in \mathbb{R}$ so

$$\begin{aligned} L^2_{a+r\widetilde{e}_0} &= (L_a + rL_{\widetilde{e}_0})^2 = L_a^2 + r^2 L^2_{\widetilde{2}_0} + r(L_a L_{\widetilde{e}_0} + L_{\widetilde{e}_0} L_a) \\ R^2_{a+r\widetilde{e}_0} &= (R_a + rR_{\widetilde{e}_0})^2 = R_a^2 + r^2 R^2_{\widetilde{e}_0} + r(R_a R_{\widetilde{e}_0} + R_{\widetilde{e}_0} R_a). \end{aligned}$$

By the doubly pure case $L_a^2 = R_a^2$, also $L^2_{\widetilde{e}_0} = R^2_{\widetilde{e}_0} = -I$ by lemma 2.1 $L_a L_{\widetilde{e}_0} + L_{\widetilde{e}_0} L_a = R_a R_{\widetilde{e}_0} + R_{\widetilde{e}_0} R_a = 0$ therefore $R^2_{a+r\widetilde{e}_0} = L^2_{a+r\widetilde{e}_0} = L_a^2 - r^2 I = R_a^2 - r^2 I$.

Q.E.D.

Remark: For $0 \neq a$ pure L_a and R_a are quite different because the center of $\mathbb{A}_n$ is $\mathbb{R}e_0$.

Notation: For a and b non–zero pure elements.

$$\mathcal{A} = L_a^2 + R_b^2 \quad \text{and} \quad S = (a, -, b) = R_b L_a - L_a R_b.$$

are linear transformations $\mathcal{A}, S : \mathbb{A}_n \to \mathbb{A}_n$.

Notice that $\mathcal{A}$ is the sum of two symmetric non–positive definite linear transformations so $\mathcal{A}$ is also symmetric non–postive definite i.e. $L_a^2 \leq 0$ and $R_b^2 \leq 0$ implies that $\mathcal{A} = L_a^2 + R_b^2 \leq 0$ and $S = [R_b, L_a]$ is skew–symmetric.

Theorem 2.3 For a and b (non–zero) pure elements in $\mathbb{A}_n$

$$L^2_{(a,b)} : \mathbb{A}_{n+1} \to \mathbb{A}_{n+1}$$

is given by

$$L^2_{(a,b)} = \begin{pmatrix} \mathcal{A} & -S \\ S & \mathcal{A} \end{pmatrix}$$

i.e. $L^2_{(a,b)}(x,y) = (\mathcal{A}(x) - S(y), \mathcal{A}(y) + S(x))$.

Proof: (By direct calculation).

$$\begin{aligned}
(a,b)[(a,b)(x,y)] &= (a,b)(ax - \overline{y}b, ya + b\overline{x}) \\
&= (a(ax) - a(\overline{y}b) - \overline{(ya + b\overline{x})}b, (ya + b\overline{x})a + b\overline{(ax - \overline{y}b)}) \\
&= (a(ax) - a(\overline{y}b) + (a\overline{y})b - (x\overline{b})b, (ya)a + (b\overline{x})a - b(\overline{x}a) - b(\overline{b}y)) \\
&= (L^2_a(x) + (a, \overline{y}, b) + R^2_b(x), R^2_a(y) + (b, \overline{x}, a) + L^2_b(y)) \\
&= (\mathcal{A}(x) - (a,y,b), \mathcal{A}(y) + (a,x,b)) \\
&= (\mathcal{A}(x) - S(y), \mathcal{A}(y) + S(x)).
\end{aligned}$$

Q.E.D. **Corollary 2.4** For r and s real numbers with $r^2 + s^2 = 1$ and a and b pure elements in $\mathbb{A}_n$ we have that

1) $L^2_{(ra-sb,sa+rb)} = L^2_{(a,b)}$ in particular $L^2_{(-b,a)} = L^2_{(a,b)}$

2) $L^2_{(ra+sb,sa-rb)} = L^2_{(b,a)}$ in particular $L^2_{(a,-b)} = L^2_{(b,a)}$

Proof: 1)

$$\begin{aligned}
L^2_{ra-sb} &= (rL_a - sL_b)^2 = r^2L^2_a + s^2L^2_b - rs(L_aL_b + L_bL_a) \\
L^2_{sa+rb} &= (sL_a + rL_b)^2 = s^2L^2_a + r^2L^2_b + sr(L_aL_b + L_bL_a).
\end{aligned}$$

Thus $L^2_{ra-sb} + L^2_{sa+rb} = (r^2+s^2)L^2_a + (r^2+s^2)L^2_b = L^2_a + R^2_b := \mathcal{A}$.

On the other hand for all $x \in \mathbb{A}_n$

$$\begin{aligned}
(ra - sb, x, sa + rb) &= (ra, x, sa) + (ra, x, rb) - (sb, x, sa) - (sb, x, rb) \\
&= 0 + r^2(a,x,b) - s^2(b,x,a) - 0 \\
&= (r^2 + s^2)(a,x,b) \\
&:= S(x)
\end{aligned}$$

by Theorem 2.2 we are done with 1).

To show 2) we observe that $\mathcal{A} = L^2_a + R^2_b = L^2_b + R^2_a$ and

$$S = (a, -, b) = -(b, -a) \ \text{ so } \ L^2_{(b,a)} = \begin{pmatrix} \mathcal{A} & S \\ -S & \mathcal{A} \end{pmatrix}$$

and the prove is analogous to case 1).

Q.E.D.

Corollary 2.5 For $\alpha = (a, b) \in \mathbb{A}_n \times \mathbb{A}_n = \mathbb{A}_{n+1}$ doubly pure we have that:

$$L_\alpha L_{\widetilde{\alpha}} + L_{\widetilde{\alpha}} L_\alpha = 0$$

and

$$R_\alpha R_{\widetilde{\alpha}} + R_{\widetilde{\alpha}} R_\alpha = 0.$$

Proof: Suppose that r and s are in $\mathbb{R}$ and $r^2 + s^2 = 1$ so $r\alpha + s\widetilde{\alpha} = r(a, b) + s(-b, a) = (ra - sb, rb + sa)$ then $L^2_{r\alpha+s\widetilde{\alpha}} = (r^2 + s^2)L^2_\alpha$ by Corollary 2.4.

On the other hand

$$\begin{aligned} L^2_{r\alpha+s\widetilde{\alpha}} &= (rL_\alpha + sL_{\widetilde{\alpha}})^2 = r^2 L^2_\alpha + s^2 L^2_{\widetilde{\alpha}} + rs(L_\alpha L_{\widetilde{\alpha}} + L_{\widetilde{\alpha}} L_\alpha) \\ &= (r^2 + s^2)L^2_\alpha \quad \text{because} \quad L^2_\alpha = L^2_{\widetilde{\alpha}}. \end{aligned}$$

so $L_\alpha L_{\widetilde{\alpha}} + L_{\widetilde{\alpha}} L_\alpha = 0$.

Similarly $R_\alpha R_{\widetilde{\alpha}} + R_{\widetilde{\alpha}} R_\alpha = 0$.

Q.E.D.

Remark Notice that Lemma 2.1 and Corollary 2.5 give us a way to define *Cuaternionic Structures* on $\mathbb{A}_n$ for n bigher than 2 via alternative elements of norm one, because for such an element a we have also that $L^2_a = R^2_a = -I$. (See [Po]).

Now the theory of Schur complement for partitioned matrices ([Z] Chapter 2 and [H-J]) tell us that.

If $\mathcal{A}$ is an invertible matrix then

$$\det(L^2_{(a,b)}) = \det(\mathcal{A}) \det(\mathcal{A} + S\mathcal{A}^{-1}S)$$

On the other hand $\mathcal{A}$ is symmetric definite non–positive i.e. $\mathcal{A} \le 0$ because also $L^2_a \le 0$ and $R^2_b \le 0$ then $-\mathcal{A} \ge 0$, $-L^2_a \ge 0$ and $-R^2_b \ge 0$ and $\det(-\mathcal{A}) = \det(-L^2_a - R^2_b) \ge \det(-L^2_a) + \det(-R^2_b)$ but $\mathcal{A}$, L^2_a and R^2_b are matrices of order 2^n so $\det(\mathcal{A}) \ge \det(L_a)^2 + \det(R_b)^2$.

Therefore if $\det(\mathcal{A}) = 0$ then $\det(L_a) = 0$ and $\det(R_b) = 0$ and $\mathcal{A}$ is invertible if, either, L_a or R_b is invertible.

Suposse now that $-\mathcal{A} > 0$. i.e. L_a or R_b are invertible then $-\mathcal{A}^{-1} > 0$ and since $S^T = -S$. $(S\mathcal{A}^{-1}S)^T = S^T(\mathcal{A}^{-1})^T S^T = S\mathcal{A}^{-1}S$ so $S\mathcal{A}^{-1}S$ is

symmetric and $\langle S\mathcal{A}^{-1}S(x), x\rangle = \langle -\mathcal{A}^{-1}S(x), S(x)\rangle \geq 0$ for all $x \in \mathbb{A}_n$, because $-\mathcal{A}^{-1} > 0$, so $(S\mathcal{A}^{-1}S) \geq 0$ and $(S\mathcal{A}^{-1}S)(x) = 0$ if and only if $S(x) = 0$.

We resume this discussion in

Theorem 2.6 For $(a, b) \in {}_0\mathbb{A}_n \times {}_0\mathbb{A}_n = \widetilde{\mathbb{A}}_{n+1}$ for $n \geq 3$. If L_a or R_b are invertible and $\lambda = -1$ is no an eigenvalue of $(\mathcal{A}^{-1}S)^2$ then $L_{(a,b)}$ is invertible in $\mathbb{A}_{n+1}$.

Proof: We already show that L_a or R_b invertible implies that $\mathcal{A}$ is invertible.

Now $L_{(a,b)}$ is invertible if and only if $L^2_{(a,b)}$ is invertible and, by Schur complement, that happen if and only if $\det(\mathcal{A} + S\mathcal{A}^{-1}S) \neq 0$ but $\det(\mathcal{A} + S\mathcal{A}^{-1}S) = \det(\mathcal{A}^{-1})\ \det(I + (\mathcal{A}^{-1}S)^2)$ so if $\lambda = -1$ is an eigenvalue of $(\mathcal{A}^{-1}S)^2$ then $\det(I + (\mathcal{A}^{-1}S)^2) = 0$.

Q.E.D.

This theorem is useful when $\mathcal{A}$ has a simple expression for instance when a and b are alternative elements so $\mathcal{A} = L_a^2 + R_b^2 = (a^2 + b^2)I$ so $\mathcal{A}^{-1} = (a^2 + b^2)^{-1}I$ and $\mathcal{A}^{-1}S = (a^2 + b^2)^{-1}S$ then $L^2_{(a,b)}$ is singular if there exists $x \neq 0$ in $\mathbb{A}_n$ such that $S^2(x) = -(a^2 + b^2)^2x$. (See [8] § 2).

§ 3. The spectrum of a doubly pure element.

Let $a \in \mathbb{A}_n$ be a doubly pure element with $||a|| \neq 0$

By the diagonalization theorems of skew–symmetric matrices [Pr] we know that there exists an orthogonal basis with respect to which L_a has the form

$$\mathrm{diag}\,(\wedge_1, \wedge_2, \cdot, \cdot, \cdot, \wedge_k, 0\cdot, \cdot, \cdot, 0)$$

where $\wedge_i = \begin{pmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{pmatrix}$ for $\lambda_i \geq 0$ in $\mathbb{R}$. So with respect to the same basis L_a^2 has the form

$$\mathrm{diag}\left(-\lambda_1^2, -\lambda_1^2, \cdot, \cdot, \cdot, -\lambda_k^2, -\lambda_k^2, 0, 0\cdot, \cdot, 0, 0\right)$$

Now, recall that $L_a(\mathbb{H}_a) = \mathbb{H}_a$ and $L_a(\mathbb{H}_a^\perp) \subset \mathbb{H}_a^\perp$.

Since $\mathbb{H}_a$ is associative we have that

$$L_a^2|_{\mathbb{H}_a} = a^2 I_{4\times 4}$$

Therefore the first two $\lambda' s$,(i.e. λ_1 andλ_2) are equal to $||a||$ so we restrict ourselves to

$$L_a, L_a^2 : \mathbb{H}_a^\perp \to \mathbb{H}_a^\perp.$$

Define for $\lambda \geq 0$ and a doubly pure with $||a|| = 1$

$$V_\lambda = \{x \in \mathbb{H}_a^\perp | a(ax) = -\lambda^2 x\}.$$

Theorem 3.1 For $0 \neq x \in V_\lambda$ the set $\{x, (-1/\lambda)(ax), (-1/\lambda)(\widetilde{ax}), \widetilde{x}\}$ is an orthogonal set in V_λ for nonzero λ and the dimension of V_λ is congruent with 0 mod 4.

Proof: If $0 \neq x \in V_\lambda$ then $\widetilde{x} \in V_\lambda$ and $a\widetilde{(ax)} = -\lambda^2 \widetilde{x}$. Also

$$a(a((-1/\lambda)(ax))) = (-1/\lambda)a(a(ax)) = (-1/\lambda)(-\lambda^2 ax) = -\lambda^2(-1/\lambda)(ax)$$

and $(-1/\lambda)(ax) \in V_\lambda$ therefore $(-1/\lambda)(\widetilde{ax}) \in V_\lambda$ and by construction $\{x, (-1/\lambda)(ax), (-1/\lambda)(\widetilde{ax}), \widetilde{x}\}$ is an orthonormal set. Now take $0 \neq y \in (\mathbb{H}_a \oplus \{x, (1/\lambda)(ax), (-1/\lambda)(\widetilde{ax}), \widetilde{x}\})^\perp$

Therefore y is doubly pure and orthogonal to $a, \tilde{a}, x, ax, \widetilde{ax}$ and $\tilde{x}$ using that $L_a^T = -L_a$ we have that:

$$\begin{array}{rcl}
\langle ay, x\rangle & = & \langle y, -ax\rangle = 0 \quad \text{so} \quad ay \perp x. \\
\langle ay, ax\rangle & = & \langle y, -a(ax)\rangle = \langle y, -(-\lambda^2 x)\rangle = \lambda^2\langle y, x\rangle = 0 \\
\langle ay, \widetilde{ax}\rangle & = & \langle ay, -\tilde{a}x\rangle = \langle \tilde{a}(ay), x\rangle = \langle -a\widetilde{(ay)}, x\rangle \\
 & = & \langle a(ay), \tilde{x}\rangle = \langle y, a(a\tilde{x})\rangle = \langle y, -\lambda^2\tilde{x}\rangle = 0 \\
\langle ay, \tilde{x}\rangle & = & \langle -\widetilde{ay}, x\rangle = \langle \tilde{a}y, x\rangle = \langle y, -\tilde{a}x\rangle = \langle y, \widetilde{ax}\rangle = 0 \\
\text{Similarly} & & \langle \tilde{y}, x\rangle = \langle -y, \tilde{x}\rangle = 0; \langle \tilde{y}, ax\rangle = \langle y, -\widetilde{ax}\rangle = 0 \\
\langle \tilde{y}, \widetilde{ax}\rangle & = & \langle y, ax\rangle = 0
\end{array}$$

Finally $\langle \widetilde{ay}, x\rangle = \langle -\tilde{a}y, x\rangle = \langle y, \tilde{a}x\rangle = 0$ and

$$\begin{array}{rcl}
\langle \widetilde{ay}, ax\rangle & = & \langle -\tilde{a}y, ax\rangle = \langle y, \tilde{a}(ax)\rangle = \langle y, -a\widetilde{(ax)}\rangle = \\
 & = & \langle y, +\lambda^2\tilde{x}\rangle = \lambda^2\langle y, \tilde{x}\rangle = 0 \\
\langle \widetilde{ay}, \widetilde{ax}\rangle & = & \langle ay, ax\rangle = 0
\end{array}$$

Therefore for any $0 \neq y \in \mathbb{H}_a^\perp$ with $y \in \{x, (-1/\lambda)ax, (1/\lambda)x, \widetilde{ax}, \tilde{x}\}^\perp$ we have that $Span\{y, \tilde{y}, ay, \widetilde{ay}\} \cap Span\{x, (-1/\lambda)ax, (-1/\lambda)\widetilde{ax}, \tilde{x}\} = \{0\}$.

Q.E.D.

Therefore the eigenvalues of $L_a^2 : \mathbb{H}_a^\perp \to \mathbb{H}_a^\perp$ have multiplicity congruent with 0 mod 4 and

$$\mathbb{H}_a^\perp = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k} \qquad k = \frac{2^n - 4}{4} = 2^{n-2} - 1$$

Definition: The *Spectrum* of a a non–zero doubly pure element, Spec(a), is the set of $(2^{n-2} - 1)$ eigenvalues (of multiplicity congruent with 0 mod 4) of

$$-L^2_{\frac{a}{||a||}} : \mathbb{H}_a^\perp \to \mathbb{H}_a^\perp$$

.

By definition the elements in Spectra are nonnegative real numbers.

For instance if $a \neq 0$ in $\widetilde{\mathbb{A}_3}$ then

$$\text{Spec }(a) = \{1\}.$$

Recall that $a \in \mathbb{A}_n$ is alternative if $a(ax) = a^2x$ for all x so for $0 \neq a \in \mathbb{A}_n$ alternative.

$$\text{Spec }(a) = \{1, 1, \ldots, 1\}.$$

Also L_a is non–singular if and only if $0 \notin \text{Spec}(a)$.

Example: Take e_1 and e_2 basic elements in $\mathbb{A}_3$.

Define $a = (e_1, e_2) \in \mathbb{A}_4$ so $||a|| = \sqrt{2}$ and

$$\begin{aligned}(e_1, e_2)(-e_4, e_7) &= (-e_1e_4 + e_7e_2, e_7e_1 + e_2e_4) = (-e_5 + e_5, -e_6 + e_6) \\ &= (0, 0)\end{aligned}$$

so $0 \in \text{Spec}(a)$ and $L_{(e_1,e_2)}$ is non–invertible.

Also

$$\begin{aligned}(e_1, e_2)(e_2, e_1) &= (2e_1e_2, 0) \qquad \text{and} \\ L^2_{(e_1,e_2)}(e_2, e_1) &= 2(e_1(e_1e_2), e_2(e_2, e_1)) = -2(e_2, e_1).\end{aligned}$$

and $1 \in \text{Spec}(a)$.

The spectrum of a has three elements we already calculate two $\{0, 1\}$.

Consider

$$\begin{aligned}(e_1, e_2)(e_7, -e_4) &= (e_1e_7 - e_4e_2, -e_4e_1 - e_2e_7) \\ &= (e_6 + e_6, e_5 + e_5) \\ &= (2e_6, 2e_5).\end{aligned}$$

so

$$\begin{aligned}L^2_{(e_1,e_2)}(e_7, -e_4) &= 2(e_1, e_2)(e_6, e_5) \\ &= 2[(e_1e_6 + e_5e_2, e_5e_1 - e_2e_6)] \\ &= 2[(-e_7 - e_7, e_4 + e_4)] \\ &= 2[(-2e_7.2e_4)] \\ &= -4(e_7, -e_4).\end{aligned}$$

Since $|(e_1, e_2)| = \sqrt{2}$ then $L^2_{\frac{a}{||a||}} = (\frac{1}{\sqrt{2}})^2 L^2_{(e_1,e_2)}$ so

$$-\frac{1}{2}L^2_{(e_1,e_2)}(e_7, -e_4) = \frac{1}{2}(4)(e_7, -e_4) = 2(e_7, -e_4)$$

so Spec$((e_1, e_2)) = \{0, 1, 2\}$.

Remarks: We can generalize the argument in last example in the following way:

Suppose that a and b are alternative elements in ${}_0\mathbb{A}_n$ $n \geq 3$ such that $|a| = |b| \neq 0$ and a orthogonal to b then

(i) $1 \in$ Spec$((a, b))$ in $\mathbb{A}_{n+1}$.

(ii) If also $(a, b)(x, y) = (0, 0)$ for some x and y in ${}_0\mathbb{A}_n$ then $2 \in$ Spec$((a, b))$ realized by (y, x) and therefore $\{0, 1, 2\} \subset$ Spec$((a, b))$.

It seems to us that (ii) is an expresion of a"Mirror Symmetry" sourrended by the zero divisors in $\mathbb{A}_{n+1}$:

"If (x, y) left 0 in the spectrum of (a, b) then (y, x) leaves 2 in the spectrum of (a, b).

"If (x, y) left 1 in the spectrum of (a, b) then (y, x) leaves 1 in the spectrum of (a, b).

This "Mirror Symmetry" is "Broken" in the Singular set defined by The zero set of the Hopf map (see [M3])

$$\{(x, y)|xy = 0, ||x|| = ||y|| = 1\}$$

Because $(x, y)(y, x) = (2xy, ||y||^2 - ||x||^2)$ for x and y pure non zero elements in $\mathbb{A}_n$, "The Hopf construction map" in terms of the algebra $\mathbb{A}_{n+1}$.

Remark: For r and s in $\mathbb{R}$ with $r^2 + s^2 = 1$ and (a, b) doubly pure in $\mathbb{A}_{n+1}$

$$\phi(a, b) = (ra \mp sb, sa \pm rb)$$

This define an $O(2)$-action on $\widetilde{\mathbb{A}}_{n+1}$ and by Corollary 2.4 we have that

$$Spec(\phi(a, b)) = Spec((a, b))$$

Now we "measure" the lack of the Normed property in $\mathbb{A}_n$ in terms of Spectra.

Definition: An element $0 \neq a \in {}_0\mathbb{A}_n$ (i.e. a is pure) *is normed with* $x \in \mathbb{A}_n$ if $||a||||x|| = ||ax||$. We also define that a *is normed* if $||ax|| = ||a||||x||$ for all $x \in \mathbb{A}_n$.

Theorem 3.2 $0 \neq a \in {}_0\mathbb{A}_n$ is normed if and only if a is alternative i.e.

$a(ax) = a^2 x$ for all x in $\mathbb{A}_n$.

Proof: Suppose that $0 \neq a \in {}_0\mathbb{A}_n$ is alternative so

$$\begin{aligned} ||a||^2||x||^2 &= -a^2\langle x, x\rangle = \langle -a^2x, x\rangle = \langle -a(ax), x\rangle = \langle ax, ax\rangle \\ &= ||ax||^2 \qquad \text{for all} \quad x \in \mathbb{A}_n. \end{aligned}$$

Conversly suppose that $||ax|| = ||a||||x||$ for all $x \in \mathbb{A}_n$ then $\langle -a^2x, x\rangle = \langle ax, ax\rangle = \langle -a(ax), x\rangle$ and $\langle a^2x - a(ax), x\rangle = 0$ for all $x \in \mathbb{A}_n$.

But $(a^2I - L_a^2)$ is a symmetric linear transformation then all its eigenvalues are of the form $\langle (a^2I - L_a^2)(x), x\rangle$ so $a^2I - L_a^2 = 0$.

Q.E.D.

Remark: Notice that the properties for an element of being normed and alternative coincide globally but no locally i.e. we say that $a \in {}_0\mathbb{A}_n$ **alternate with** x for $x \in \mathbb{A}_n$ if $a(ax) = a^2x$ so $a \in {}_0\mathbb{A}_n$ is an alternative element if a alternates with any element in $\mathbb{A}_n$ (see[M2]).

Now if a alternate with x then a is normed with $x : ||ax||^2 = \langle ax, ax\rangle = \langle -a(ax), x\rangle = \langle -a^2x, x\rangle = ||a||^2 \cdot ||x||$.

But the converse is no necessarily true.

Example: Take $a = e_1$, $b = e_2$ and $\widetilde{e}_0 = e_4$ in $\mathbb{A}_3$. Define $\alpha = (e_1, e_2)$ and $\varepsilon = (\widetilde{e}_0, 0)$ in $\mathbb{A}_3 \times \mathbb{A}_3 = \mathbb{A}_4$. Since ε is alternative element in $\mathbb{A}_4$ we have that $||\alpha\varepsilon||^2 = \langle \alpha\varepsilon, \alpha\varepsilon\rangle = \langle -(\alpha\varepsilon)\varepsilon, \alpha\rangle = \langle -\alpha\varepsilon^2, \alpha\rangle = ||\alpha||^2||\varepsilon||^2 = ||\alpha||^2$ and α is normed with ε.

On the other hand using th.2.3

$$\begin{aligned} L_\alpha^2(\varepsilon) &= ((L_{e_1}^2 + R_{e_2}^2)(\widetilde{e}_0) - 0, 0 + (e_1, \widetilde{e}_0, e_2)) \\ &= ((-2I)(e_4), (e_1e_4)e_2 - e_1(e_4e_2)) \\ &= (-2e_4, e_5e_2 - e_1(-e_6)) \\ &= (-2e_4, -e_7 - e_7) \\ &= (-2e_4, -2e_7) \\ &= -2\varepsilon - 2e_{15} \qquad \text{in} \ \ \mathbb{A}_4. \end{aligned}$$

But $\alpha^2\varepsilon = -2\varepsilon$ so $(\alpha, \alpha, \varepsilon) = \alpha^2\varepsilon - L_\alpha^2(\varepsilon) = 2e_{15}$ and α does not alternate with ε.

Proposition 3.3 For $a \in \widetilde{\mathbb{A}}_n$ with $a \neq 0$ and $\lambda \in \mathrm{Spec}(a)$.

Suppose that $0 \neq x \in V_\lambda \subset \mathbb{H}_a^\perp \subset \widetilde{\mathbb{A}}_n$ then

(1)$||ax|| = \lambda||a||||x||$.

(2) If x is an alternative element then $\lambda = 1$.

Proof: (1) Assuming that $||a|| = 1$ we have that

$$a(ax) = -\lambda^2 x.$$

so

$$||ax||^2 = \langle ax, ax\rangle = -\langle a(ax), x\rangle = \langle \lambda^2 x, x\rangle = \lambda^2||x||^2$$

.

Therefore in general we have that

$$||ax|| = \lambda||ax||$$

.

If $0 \neq x$ is an alternative element then $||yx|| = ||y||||x||$ for all y (th 3.2).Thus if $x \in V_\lambda$, then by (1), $\lambda = 1$, and we are done with (2).

Q.E.D.

Remark There are elements $0 \neq a \in \widetilde{\mathbb{A}}_n$ with $1 \in \mathrm{Spec}(a)$ and a non–alternative (See example above).

Moreover now we prove that for each nonzero $a \in \widetilde{\mathbb{A}}_n$,

$$1 \in \ \text{Spec} \ (a)$$

For this we need:

Lemma 3.4 For $a \neq 0$ in $\widetilde{\mathbb{A}}_n \quad n \geq 4$ there exists $x \neq 0$ in $\mathbb{H}_a^\perp$ such that $||ax|| = ||a||||x||$.

Proof: Recall that if $0 \neq y \in \mathbb{A}_n$ is alternative then $||ay|| = ||a||||y||$. (th 3.2).Thus If $0 \neq y \in \mathbb{H}_a^\perp$ for some y alternative we are done.

On the other hand by dimensional reasons (recall dim $\mathbb{H}_a = 4$ and any canonical basic element is alternative) no all alternative element belongs to $\mathbb{H}_a$ so there exists $0 \neq y \in \mathbb{A}_n$ alternative element with $y = (y_1 + y_2) \in \mathbb{H}_a \oplus \mathbb{H}_a^\perp$ and $y_2 \neq 0$ in $\mathbb{H}_a^\perp$.

But $||ay|| = ||a||||y||$ implies that

$$\langle ay, ay\rangle = \langle a, a\rangle\langle y, y\rangle$$

But

$$\begin{aligned}\langle ay, ay\rangle &= \langle a(y_1+y_2), a(y_1+y_2)\rangle\\ &= \langle ay_1+ay_2, ay_1+ay_2\rangle\\ &= \langle ay_1, ay_1\rangle + \langle ay_2, ay_2\rangle + 2\langle ay_1, ay_2\rangle\end{aligned}$$

But $\langle ay_1, ay_2\rangle = -\langle a(ay_1), y_2\rangle = -\langle a^2y_1, y_2\rangle = 0$ and $\langle ay_1, ay_1\rangle = \langle a, a\rangle\langle y_1, y_1\rangle = ||a||^2||y_1||^2$ because $y_1 \in \mathbb{H}_a$ wich is associative and $y_2 \perp y_1$; therefore

$$\begin{aligned}\langle ay, ay\rangle &= \langle a, a\rangle\langle y_1, y_1\rangle + \langle ay_2, ay_2\rangle \qquad \text{and}\\ \langle a, a\rangle\langle y, y\rangle &= \langle a, a\rangle\langle y_1+y_2, y_1+y_2\rangle\\ &= \langle a, a\rangle\langle y_1, y_1\rangle + \langle a, a\rangle\langle y_2, y_2\rangle\end{aligned}$$

because $y_1 \perp y_2$.

Cancelling out we have $\langle a, a\rangle\langle y_2, y_2\rangle = \langle ay_2, ay_2\rangle$ and $||a||||y_2|| = ||ay_2||$ for $y_2 \neq 0$ in $\mathbb{H}_a^{\perp}$

Q.E.D.

Remark: We can also use the construction of bilinear maps as in Adem (see [A],[L] and [Sh]) to prove this lemma because any non–zero double pure element belongs to a alternative subalgebra of $\mathbb{A}_n$ with normed product (see [M2]).

Theorem 3.5 For $0 \neq a \in \mathbb{A}_n$ $n \geq 3$ with a doubly pure

$$1 \in \text{ Spec}(a).$$

Proof: Suppose that $1 \notin \text{Spec}(a)$ then $a(ax) \neq a^2x$ for all (non-zero) $x \in \mathbb{H}_a^{\perp}$ then $a(ax) - a^2x \neq 0$ for all (non-zero) $x \in \mathbb{H}_a^{\perp}$. Now $L_a^2(\mathbb{H}_a) \subset \mathbb{H}_a$ and $L_a^2(\mathbb{H}_a^{\perp}) \subset \mathbb{H}_a^{\perp}$ so we have that $L_a^2|_{\mathbb{H}_a^{\perp}}$ and $(L_a^2 - a^2I)|_{\mathbb{H}_a^{\perp}}$ are symmetric linear transformations.

Therefore $\langle a(ax) - a^2x, x\rangle \neq 0$ for all non-zero $x \in \mathbb{H}_a^{\perp}$ and $\langle ax, ax\rangle \neq |a|^2\langle x, x\rangle$ and $||ax|| \neq ||a||||x||$ for all non-zero $x \in \mathbb{H}_a^{\perp}$ which contradicts the previous lemma 3.4.

Q.E.D.

Corollary 3.6 For a nonzero element in $\mathbb{A}_n$ $n \geq 3$ we have that:

$$dimKer(a^2I - L_a^2) \geq 8.$$

Proof. Since any associator vanish when one of the entries is real,the question can be reduced to the case a is a pure element.

On the other hand by Corollary 1.5 in [M2] is known that for b double pure element

$$(b, \widetilde{e}_0, x) + (\widetilde{e}_0, b, x) = 0$$

for all x in $\mathbb{A}_n$.

Recalling that $\widetilde{e}_0$ is an alternative element, we can reduce the question to tha case a double pure element,so by Theorem 3.5. $dimV_\lambda \geq 4$ for $\lambda = 1$ and

$$dim(\mathbb{H}_a \oplus V_1) \geq 8$$

Q.E.D.

§ 4. The zero divisors in $\mathbb{A}_n$ $\quad n \geq 4$.

In this section $0 \neq a = (a_1, a_2) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1} = \mathbb{A}_n$ and $n \geq 4$

Definition: *a is zero divisor if* there exists $x \neq 0$ in $\mathbb{A}_n$ such that $ax = 0$ i.e.

$$\text{Ker } L_a \neq \{0\}.$$

Since $||ax|| = ||xa||$ for all x in $\mathbb{A}_n$ (see [M1]) we have that $ax = 0$ if and only if $xa = 0$ so left and right zero divisors concide.

Also $||ax|| = ||\overline{a}x||$ (see [M1]) then if $x \neq 0$ $\; t_n(a)x = (a+\overline{a})x = ax+\overline{a}x = 0$ if $ax = 0$; but $t_n(a)$ is real number so $ax = 0 \Rightarrow t_n(a) = 0$ so any zero divisor is a pure element.

Actually

Lemma 4.1 Any zero divisor in $\mathbb{A}_n$ is double pure.

Proof: By Corollary 2.4 for $a \in {}_0\mathbb{A}_n$

$$\text{Ker } L_a = \text{ Ker } L_a^2 = \text{ Ker } L_{\widetilde{a}}^2 = \text{ Ker } L_{\widetilde{a}}$$

If $\{0\} \neq \text{Ker}L_a$ then $\{0\} \neq \text{Ker } L_{\widetilde{a}}$ and $\widetilde{a} = (-a_2, a_1)$ is a zero divisor so $\widetilde{a}$ is pure in $\mathbb{A}_n$ and a_2 is pure in $\mathbb{A}_{n-1}$ then a is doubly pure.

Q.E.D.

Notice that $ax = 0$ if and only if $||a||^{-1}ax = 0$ so being a zero divisor is independent of the norm. On the other hand by Corollay 2.4(1) if a is a zero divisor and r and s in $\mathbb{R}$ with $r^2 + s^2 = 1$ then $L_a^2 = L_{(a_1,a_2)}^2 = L_{(ra_1 - sa_2, sa_1 + ra_2)}^2 = L_{ra+s\widetilde{a}}^2$ then $(ra + s\widetilde{a})$ is a zero divisor (in particular $\widetilde{a}$ is a zero divisor) and for any $b \neq 0$ in $\mathbb{H}_a$ doubly pure we have that b is a zero divisor and $\text{Ker}L_a = \text{Ker}L_b$. Because

$$\begin{aligned} L_{ra+s\widetilde{a}}^2 &= r^2 L_a^2 + s^2 L_{\widetilde{a}} + rs(L_a L_{\widetilde{a}} + L_{\widetilde{a}} L_a) \\ &= (r^2 + s^2) L_{\widetilde{a}}^2 + 0 \qquad \text{(Corollary 2.5)} \\ &= L_a^2. \end{aligned}$$

So if $||a|| = 1$ then $b = ra + s\widetilde{a}$ with $r^2 + s^2 = 1$ and $\text{Ker } L_a = \text{Ker}L_b$.

Lemma 4.2 Let $a = (a_1, a_2)$ be in $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1} = \mathbb{A}_n$ with $a \neq 0$ $\;.a$ is a zero divisor if and only if $\hat{a} := (a_2, a_1)$ is a zero divisor. Moreover $ax = 0$ if

and only if $\hat{a}\hat{x} = 0$.

Proof: Put $x = (x_1, x_2)$ so $\hat{x} = (x_2, x_1)$ in ${}_0\mathbb{A}_{n-1} \times {}_0\mathbb{A}_{n-1}$ calculating we have:

$$\begin{aligned} ax &= (a_1, a_2)(x_1, x_2) = (a_1x_1 + x_2a_2, x_2a_1 - a_2x_1) := (c, d) \\ \hat{a}\hat{x} &= (a_2, a_1)(x_2, x_1) = (a_2x_2 + x_1a_1, x_1a_2 - a_1x_2) = (\overline{c}, -\overline{d}) \end{aligned}$$

therefore $c = d = 0$ in $\mathbb{A}_{n-1}$ if and only if $ax = \hat{a}\hat{x} = 0$.

Q.E.D.

Remark: Notice that lemma 4.1 said $\mathrm{Ker}L_a = \mathrm{Ker}L_{\widetilde{a}}$ (equality) while lemma 4.2 said that $\mathrm{Ker}L_a \cong \mathrm{Ker}L_{\hat{a}}$ (isomorphism), induced by

$${}_0\mathbb{A}_{n-1} \times {}_0\mathbb{A}_{n-1} \xrightarrow{\wedge} {}_0\mathbb{A}_n \times {}_0\mathbb{A}_n$$

given by $(x_1, x_2) \mapsto (x_2, x_1)$ which is a one–to one correspondence.

Now $0 \neq a \in \mathbb{A}_n$ is a zero divisor if and only if ra is a zero divisor for all $r \in \mathbb{R}$, $r \neq 0$ so we restrict ourselves to the study of zero divisor of norm one.

Using lemma 1.1 in §1 we can construct zero divisors in the following way:

Proposition 4.3. For $0 \neq a \in \mathbb{A}_n$, $n \geq 3$, and a doubly pure we have:

1) If $b \in \mathbb{H}_a^{\perp}$ and $|a| = |b| \neq 0$ then (a, b) is a zero divisor in $\mathbb{A}_{n+1}$.

2) $(a, \widetilde{a})$ and $(a, -\widetilde{a})$ are zero divisors in $\mathbb{A}_{n+1}$.

Proof:

1) For $b \in \mathbb{H}_a^{\perp}$ with $|a| = |b| \neq 0$ we have that $(a, b)(\widetilde{a}, \widetilde{b}) = (a\widetilde{a} - \widetilde{\widetilde{b}}b, \widetilde{b}a + b\widetilde{\widetilde{a}}) = (a\widetilde{a} + \widetilde{b}b, \widetilde{b}a - b\widetilde{a})$ in $\mathbb{A}_{n+1}$. By lemma 1.1. (2) $a\widetilde{a} + \widetilde{b}b = -||a||^2\widetilde{e}_0 + ||b||^2\widetilde{e}_0 = 0$ in $\mathbb{A}_n$. By lemma 1.1. (6) $\widetilde{b}a = b\widetilde{a}$ because $b \in \mathbb{H}_a^{\perp}$ in $\mathbb{A}_n$.

2) Since $n \geq 3$, $\mathbb{H}_a^{\perp} \neq \{0\}$. Taking $0 \neq x \in \mathbb{H}_a^{\perp} \subset \mathbb{A}_n$

$$(a, \widetilde{a})(x, -\widetilde{x}) = (ax - \widetilde{x}\widetilde{a}, -\widetilde{x}a - \widetilde{a}x) \ \text{ in } \ \mathbb{A}_{n+1}.$$

By lemma 1.1. (5) $ax = \widetilde{x}\widetilde{a}$ and by lemma 1.1 (3) $-\widetilde{x}a - \widetilde{a}x = \widetilde{xa} + \widetilde{ax} = \widetilde{xa + ax} = 0$ because $a \perp x$.

Similarly

$$(a, -\widetilde{a})(x, \widetilde{x}) = (ax - \widetilde{x}\widetilde{a}, \widetilde{x}a + \widetilde{a}x) = (0, 0) \quad \text{in} \ \ \mathbb{A}_{n+1}$$

Q.E.D.

Remark: Notice that this proposition is telling us that any non–zero *doubly pure* element in $\mathbb{A}_n$ $(n \geq 3)$ is the component of a zero divisor in $\mathbb{A}_{n+1}$.

We will prove that this is true also for non–zero *pure* elements in $\mathbb{A}_n$.

Theorem 4.4. For $0 \neq a \in \mathbb{A}_n$, $(n \geq 3)$ a doubly pure element of norm one and $0 \neq \lambda \in \mathrm{Spec}(a)$ we have that $(a, \pm\lambda\widetilde{e}_0)$ are zero divisors in $\mathbb{A}_{n+1}$.

Proof: By definition of the spectrum of a

$$a(ax) = -\lambda^2 x \qquad \text{for some} \qquad 0 \neq x \in \mathbb{H}_a^{\perp}$$

Then in $\mathbb{A}_{n+1}$

$$\begin{aligned}
(a, \lambda\widetilde{e}_0)(ax, -\lambda\widetilde{x}) &= (a(ax) - \lambda^2\widetilde{x}\widetilde{e}_0, -\lambda\widetilde{x}a + \lambda\widetilde{e}_0(\overline{ax})) \\
&= (-\lambda^2 x - \lambda^2\widetilde{\widetilde{x}}, \lambda(\widetilde{xa}) - \lambda(\widetilde{xa})) \\
&= (0, 0)
\end{aligned}$$

Similarly

$$\begin{aligned}
(a, -\lambda\widetilde{e}_0)(ax, \lambda\widetilde{x}) &= (a(ax) + \lambda^2\widetilde{x}\widetilde{e}_0, \lambda\widetilde{x}a - \lambda\widetilde{e}_0(\overline{ax})) \\
&= (-\lambda^2 x - \lambda^2\widetilde{\widetilde{x}}, -\lambda\widetilde{xa} + \lambda(\widetilde{xa}) \\
&= (0, 0).
\end{aligned}$$

Notice also that $(a, \pm\lambda\widetilde{e}_0)(-1/\lambda ax, \pm\widetilde{x}) = (0, 0)$.

Q.E.D.

From this we derive the important:

Corollary 4.5 For $0 \neq \alpha \in \mathbb{A}_n \, (n \geq 3)$ with $|\alpha| = 1$ and α pure element there exists β in $\mathbb{A}_n$ with $\alpha \perp \beta$ and $|\alpha| = |\beta| = 1$ such that (α, β) is a zero divisor in $\mathbb{A}_{n+1}$.

Proof: If α is doubly pure we are done by Proposition 4.3. Suppose that α is pure element (no double pure) then there exist $a \in \mathbb{A}_n$ doubly pure element and r and s real numbers with $r^2 + s^2 = 1$ and $s \neq 0$ such that $\alpha = ra + s\widetilde{e}_0$ and $|a| = |\alpha| = 1$.

By Corollary 2.4 (1) $L^2_{(a, -\widetilde{e}_0)} = L^2_{(ra + s\widetilde{e}_0, sa - r\widetilde{e}_0)}$. Therefore making $\beta = sa - r\widetilde{e}_0$ we have that

$$L^2_{(a, -\widetilde{e}_0)} = L^2_{(\alpha, \beta)}$$

Now by theorem 3.5 $\lambda = 1 \in \mathrm{Spec}(a)$ and by theorem 4.4 $(a, -\widetilde{e}_0)$ is a zero divisor in $\mathbb{A}_{n+1}$ so (α, β) is a zero divisor in $\mathbb{A}_{n+1}$.

By construction $|\alpha| = |\beta| = 1$ and $\alpha \perp \beta$.

Q.E.D.

Example: We know that 2 is in $Spec((e_1, e_2))$ in $\mathbb{A}_4$ so if $a = (\sqrt{2})^{-1}(e_1, e_2)$ and $\widetilde{e}_0 = e_8$ in $\mathbb{A}_4$,according with Theorem 4.4 we have that

$$(a, 2e_8) = \sqrt{2}^{-1}(e_1 + e_{10}) + 2e_{16}$$

is a zero divisor in $\mathbb{A}_5$.

Remarks: Notice that, in contrast, with the case $n = 4$ where the zero divisors must have coordinates in $\mathbb{A}_3$ of equal norm (see [M1]) this is no the case for $\mathbb{A}_5$.(Besides the "trivial" cases when one of the coordinates is a zero divisor and the other is equal to 0 in $\mathbb{A}_4$.)

Therefore the zero divisors in $\mathbb{A}_5$ are "very far" to be described as in $\mathbb{A}_4$ where they can be identified with $V_{7,2}$ the Stiefel Manifold of 2 frames in $\mathbb{R}^7$.

But also, Corollary 4.5 is telling us that the set of zero divisors in $\mathbb{A}_{n+1}$ has some subset wich can be describe in terms of the Stiefel Manifold $V_{2^n-1,2}$. for $n \geq 3$.

§5.Zero divisors and Stiefel manifolds.

Definition: $\alpha = (a,b) \in {}_0\mathbb{A}_n \times {}_0\mathbb{A}_n = \widetilde{\mathbb{A}}_{n+1}$ is a *Stiefel element* if $||a|| = ||b|| \neq 0$ and $a \perp b$ in ${}_0\mathbb{A}_n$.

Definition: $\alpha = (a,b) \in \widetilde{\mathbb{A}}_n \times \widetilde{\mathbb{A}}_n \subset \widetilde{\mathbb{A}}_{n+1}$ is a *Non–trivial element* if $||a|| = ||b|| \neq 0$ and $b \in \mathbb{H}_a^{\perp}$ i.e. $a \perp b$ and $\widetilde{a} \perp b$.

Clearly any non–trivial element in $\widetilde{\mathbb{A}}_{n+1}$ is a Stiefel element in $\widetilde{\mathbb{A}}_{n+1}$.

Lemma 5.1. If $\alpha = (a,b)$ and $\chi = (x,y)$ are in ${}_0\mathbb{A}_n \times {}_0\mathbb{A}_n = \widetilde{\mathbb{A}}_{n+1}$ then

$$\langle \alpha, \chi \rangle = \langle a, x \rangle + \langle b, y \rangle$$

Proof:

$$\begin{aligned} \alpha\chi + \chi\alpha &= (ax + yb, ya - bx) + (xa + by, bx - ya) \\ &= (ax + xa + by + yb, 0) \end{aligned}$$

so

$$-2\langle \alpha, \chi \rangle = \alpha\chi + \chi\alpha = (-2\langle a, x \rangle - 2\langle y, b \rangle, 0)$$

Q.E.D.

Theorem 5.2 $\alpha = (a,b)$ is a Stiefel element in $\widetilde{\mathbb{A}}_{n+1}$ if and only if $(\alpha, \hat{\alpha})$ is a non–trivial element in $\widetilde{\mathbb{A}}_{n+2}$.

Proof: Recall that if $\alpha = (a,b)$ then $\hat{\alpha} = (b,a)$. By lemma 5.1 $\langle \alpha, \hat{\alpha} \rangle = \langle a, b \rangle + \langle b, a \rangle$ so $\alpha \perp \hat{\alpha}$ in $\widetilde{\mathbb{A}}_{n+1}$ if and only if $a \perp b$ in ${}_0\mathbb{A}_n$.

Also by lemma 5.1 $\langle \widetilde{\alpha}, \hat{\alpha} \rangle = \langle (-b, a), (b, a) \rangle = -||b||^2 + ||a||^2$ so $\widetilde{\alpha} \perp \hat{\alpha}$ in $\widetilde{\mathbb{A}}_{n+1}$ if and only if $||a|| = ||b|| \neq 0$.

Q.E.D.

Theorem 5.3 The set of non–trivial elements in $\widetilde{\mathbb{A}}_{n+1}$ with entries of norm 1 is the complex Stiefel manifold $W_{m,2}$ for $m = 2^{n-1} - 1$.[J].

Proof: Define $\mathcal{H}_n : \widetilde{\mathbb{A}}_n \times \widetilde{\mathbb{A}}_n \to \mathbb{C}$ = complex numbers, by $\mathcal{H}_n(x,y) = 2\langle x, y \rangle - 2i\langle \widetilde{x}, y \rangle$.

Claim: $\mathcal{H}_n$ define a Hermitian inner product in $\widetilde{\mathbb{A}}_n$.

Clearly $\mathcal{H}_n$ is $\mathbb{R}$–bilinear and

$$\begin{aligned}\overline{\mathcal{H}_n(x,y)} &= 2\langle x,y\rangle + 2i\langle \widetilde{x},y\rangle = 2\langle x,y\rangle - 2i\langle \widetilde{y},x\rangle \\ &= \mathcal{H}_n(y,x).\end{aligned}$$

Also

$$\begin{aligned}\mathcal{H}_n(\widetilde{x},y) &= 2\langle \widetilde{x},y\rangle - 2i\langle \widetilde{\widetilde{x}},y\rangle = 2\langle \widetilde{x},y\rangle + 2i\langle x,y\rangle \\ &= 2i\langle x,y\rangle - 2i^2\langle \widetilde{x},y\rangle = i\mathcal{H}_n(x,y)\end{aligned}$$

Therefore $\mathcal{H}_n$ define a Hermitian product as claimed.

By definition $\mathcal{H}_n(a,b) = 0$ if and only if $b \in \mathbb{H}_a^{\perp}$ and

$$W_{m,2} = \{(a,b) \in \widetilde{\mathbb{A}}_n \times \widetilde{\mathbb{A}}_n | \mathcal{H}_n(a,b) = 0 \ \text{ and } \ ||a|| = ||b|| = 1\}$$

for

$$m = \frac{1}{2}(2^n - 2) = 2^{n-1} - 1$$

Q.E.D.

Remark Notice that the set of Stiefel elements in $\widetilde{\mathbb{A}}_{n+1}$, with entries of norm one, in ${}_0\mathbb{A}_n$ can be seen as the real Stiefel manifold $V_{2^n-1,2}$ i.e.

$$V_{2^n-1,2} = \{(a,b) \in {}_0\mathbb{A}_n \times {}_0\mathbb{A}_n | a \perp b \ \text{ and } \ |a| = |b| = 1\}$$

Now we give a partial answer to the following:

Question: Is any Stiefel element in $\widetilde{\mathbb{A}}_{n+1}$ a zero divisor?.

Partial Answer: So far, we have seen that the following types of Stiefel elements are zero divisors.

Suppose that $\alpha = (a,b) \in \widetilde{\mathbb{A}}_{n+1}$ with $a \in {}_0\mathbb{A}_n, b \in {}_0\mathbb{A}_n$ and $|a| = |b| \neq 0$ with $a \perp b$.

Case: a **and** b **doubly pure** in $\mathbb{A}_n$.

1. If $\alpha = (a,b) \in \widetilde{\mathbb{A}}_n \times \widetilde{\mathbb{A}}_n$ is a non–trivial then $b \in \mathbb{H}_a^{\perp}$ and by §4 $(a,b)(\widetilde{a},\widetilde{b}) = (0,0)$ in $\mathbb{A}_{n+1}$.

2. Suppose that $b \in \mathbb{H}_a$.

   Since $|a| = |b| \neq 0 \quad b = ra + s\widetilde{a}$ for $r^2 + s^2 = 1$, and $a \perp b$ implies that $0 = \langle a,b\rangle = \langle a,ra\rangle + \langle a,s\widetilde{a}\rangle$,but $a \perp \widetilde{a}$ and we have that $0 = r\langle a,a\rangle = r||a||^2$, therefore $r = 0$ and $b = s\widetilde{a}$ with $s^2 = 1$ i.e. $b = \pm\widetilde{a}$.

In § 4 we show that $(a, \pm\widetilde{a})(x, \mp\widetilde{x}) = (0, 0)$ in $\mathbb{A}_{n+1}$ for all $0 \neq x \in \mathbb{H}_a^{\perp}$, so (a, b) is a zero divisor.

General case: a and b pure elements in $\mathbb{A}_n$.

In § 4 we prove that for any non–zero doubly pure element c in $\widetilde{\mathbb{A}}_n$ $\quad (c, \widetilde{e}_0)$ is a zero divisor in $\mathbb{A}_{n+1}$.

Since any pure element $a \in {}_0\mathbb{A}_n$ with $a \neq 0$ is of the form $a = rc \mp s\widetilde{e}_0$ for r, s in $\mathbb{R}$ with $r^2 + s^2 = 1$ and $|a| = |c| \neq 0$ for some c doubly pure in $\widetilde{\mathbb{A}}_n$ then $(rc \mp s\widetilde{e}_0, sc \pm r\widetilde{e}_0)$ is a zero divisor so (a, b) is a zero divisor in $\mathbb{A}_{n+1}$ for $a \in {}_0\mathbb{A}_n \quad a \neq 0$ and $b = sc \pm r\widetilde{e}_0$ if $a = rc \mp s\widetilde{e}_0$.

More generally if $0 \neq \lambda \in Spec(c)$ then $(c, \lambda\widetilde{e}_0)$ is a zero divisor in $\mathbb{A}_{n+1}$ for c a doubly pure element in $\widetilde{\mathbb{A}}_n$ with $c \neq 0$ (§4). If $a = rc \mp s\lambda\widetilde{e}_0$ with r and s in $\mathbb{R}$ such that

$$r^2 + (s\lambda)^2 = 1$$

then (a, b) is a zero divisor for $b = (s\lambda)c \pm r\widetilde{e}_0$.

Open Question:

Given $a \in {}_0\mathbb{A}_n$ with $a \neq 0$.

If $b \in {}_0\mathbb{A}_n$ with $b \perp a, |b| = |a| \neq 0$ then

Is b of one the forms described above?

Remark. Notice that if $\lambda = 0$ is in $Spec(c)$ then c is a zero divisor and $a = rc \mp s\lambda\widetilde{e}_0 = rc$ with $r^2 = 1$ so $a = \pm c$ and $b = \mp\widetilde{e}_0$ and (a, b) is a zero divisor.

REFERENCES

[A] J. Adem, *Construction of some normed maps*, Bol. Soc.Mat. Mex. 20 (1975) 59–75.

[B-D-I] Biss -Dugger-Isaksen,*Large annihilators in Cayley-Dickson algebras* RA/0511691 Nov 2005.

[Ch] Chaitin-Chatelin F,Inductive multiplication in Dickson algebras CERFACS Technical Report TR/PA/05/56.

[E-S] Eakin–Sathaye, *On automorphisms and derivations of Cayley–Dickson algebras*, Jornal of Pure and Applied Algebra, 129, 263–278 (1990).

[H-J] Horn and Jhonson, *Matrix analysis*, Cambridge University Press.

[J] James I., *Topology of Stiefel manifolds*, Cambridge University Press.

[L] Lam, K.Y., *Topological methods for studying the composition of quadratic forms*, Canadian Math. Soc. Conf. Proc. 4 173–192 (1984).

[M1] Moreno G., *The zero divisor of the Cayley–Dickson algebras over the real numbers*, Bol. Soc. Mat. Mex. (tercera serie), Vol. 4 No. 1 1998.

[M2] Moreno G., *Alternative elements in the Cayley–Dickson algebras.*Proceedings in Relativity and Field theory in Honor of Jerzy Plebansky Birkhauser 2004. math.RA/0404395.

[M3] Moreno G. *Hopf construction in higher dimensions* Bol. Soc. Mat. Mex 2004 also math.AT/0404172.

[M4] Moreno G. *Monomorphisms between Cayley–Dickson algebras.*Nonassocitive algebras and its applications Chapter 22 CRC Press, Taylor and Francis 2005.math.RA/0512516.

[Po] Porteous, I., *Topological algebra*, Cambridge University Press.

[Pr] Praslov, *Problems and theorems in linear algebra.* Translation of the A.M.S., Vol. 134.

[Sch1] Schaffer, R.D., *An introduction to non–associtive algebras*, Academic Press 1966.

[Sch2] Schaffer, R.D., *On algebras formed by the Cayley Dickson Process*, American J. of Math. 76 (1954) 435–446.

[Sh] Shapiro, D., *Product of sum of squares*, Exposition Math. (2) 235–261, (1984).

[K-Y] Khalil– Yiu P., *The Cayley–Dickson algebras: A theorem of Hurwitz and quaternions.*, Bulletin de la Societé, de Lódz, Vol. XLVIII (1997)pp. 117–169.

[Z] Zhang F., *Matrix theory*, Springer–Verlag (University Text).

Guillermo Moreno Rodríguez
Departamento de Matematicas
Centro de Investigación y de estudios Avanzados
(CINVESTAV del IPN) Apartado Postal 14–740
MEXICO D.F. MEXICO 07300
gmoreno@math.cinvestav.mx